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Mach's principle and a new gauge freedom in Brans–Dicke theory

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Abstract. The relationship between Mach's principle and the Brans–Dicke scalar–tensor theory of gravitation is discussed. A converse formulation is proposed. It is shown that the geometry of space–time determines the mass–energy content only up to a position dependent gauge transformation. This is interpreted as a scale change in the unit of mass which preserves the conservation laws. For arbitrary values of the scalar field coupling constant ω the theory is invariant only under a restricted group of gauge transformations. Complete invariance applies in the case $\omega = 0$. Furthermore it is shown that whenever the curvature scalar R vanishes the theory can be transformed into the usual form of Einstein's general relativity. Equivalence classes of solutions are defined which have the same geometry and lead to the same geodesic motion for test particles. Examples are constructed for the case of static spherical symmetry and for uniform cosmology which exhibit these properties. Some further consequences are discussed.

1. Introduction

As usually formulated, Mach's principle requires that the geometry of space–time and hence the inertial properties of every infinitesimal test particle be determined by the distribution of mass–energy throughout the universe (see, eg, Wheeler 1964). Although one of the foundation stones of Einstein's philosophy, this principle is contained only to a limited extent in general relativity (Dicke 1964). Some examples of 'non-Machian' solutions are (cf Heckmann and Schucking 1962): Minkowski space which has inertial properties but no matter; the Godel universe which contains such unphysical properties as closed time-like curves; and the closed but empty Taub model. Wheeler (1964) has suggested that these unsatisfactory solutions might be excluded by means of boundary conditions. Brans and Dicke (1961, to be referred to as BD) have argued against this possibility by considering a static massive shell. The inertial properties of test particles inside the shell are, according to general relativity, unchanged even if the mass of the shell is increased.

In the hope of extending general relativity in such a way as to incorporate Mach's principle, Brans and Dicke (BD) have proposed a theory which includes a long range scalar field interacting equally with all forms of matter (with the exception of electromagnetism). They noted, following Dirac (1938) and Sciama (1959), that the newtonian gravitational constant G is related to the mass M and radius R of the visible universe by

$$G \sim \frac{Rc^2}{M}. \tag{1.1}$$

The numbers are approximate. This suggests that G is a (scalar) function determined by the matter distribution. Their theory is formally equivalent to one previously considered by Jordan (1955). As has been shown in a recent article (O'Hanlon and Tupper 1972) the BD theory also permits non-Machian solutions of the type mentioned.

A further difficulty with the above formulation of Mach's principle becomes evident when we recall (Synge 1966) that we cannot in general define the stress–energy tensor T_{ij}, from which the geometry is to be determined, unless the geometry is already known! This primacy of geometry leads us to state the following 'converse' of Mach's principle: the geometry of space–time determines uniquely the matter content. Observationally this statement is implicitly assumed when, for example, we determine the masses of the planets and moons in the solar system by a study of their orbits (the geodesics). Clearly, almost trivially, Einstein's theory satisfies this converse formulation. However, in the BD theory, as will be shown, the geometry does not uniquely specify the energy content.

To demonstrate this we will consider a group of transformations of the energy–momentum tensor which preserve both (i) the geometry and (ii) the form of the field equations and hence the law of conservation of energy. This second condition is necessary for the geodesic motion to be derivable from the field equations (Goldberg 1962), and in order that the physical significance of the geometry be retained.

2. Brans–Dicke theory and units transformations

The conditions (i) and (ii) above suggest that what we are seeking is the group of allowed transformations of the unit of mass. Einstein's equations†

$$R_{ij}-\tfrac{1}{2}g_{ij}R = 8\pi G T_{ij} \tag{2.1}$$

are invariant under the constant scale change

$$T_{ij} \to \lambda T_{ij}$$
$$G \to \lambda^{-1}G. \tag{2.2}$$

Under (2.2) the conservation equation

$$T_i{}^j{}_{;j} = 0 \tag{2.3}$$

will remain valid in the new units. Note that this will not in general be true if λ is a function of the coordinates. In the Brans–Dicke theory the gravitational 'constant' $G = \phi^{-1}$ is a function of the coordinates and we may consider nonconstant units transformations (Dicke 1962). The field equations are given (BD) as

$$R_{ij}-\tfrac{1}{2}g_{ij}R = 8\pi\phi^{-1}T_{ij}+\omega\phi^{-2}(\phi_{,i}\phi_{,j}-\tfrac{1}{2}g_{ij}\phi_{,k}\phi^{,k})+\phi^{-1}(\phi_{;ij}-g_{ij}\Box\phi) \tag{2.4}$$

$$\Box\phi = \frac{8\pi}{3+2\omega}T \tag{2.5}$$

where $\Box\phi \equiv g^{ij}\phi_{;ij}$ and $T \equiv g^{ij}T_{ij}$. The conservation law (2.3) is also valid and follows from the form of the equations (2.4), (2.5). These are obviously invariant under the scale change (2.2). Instead we consider the generalized transformation

$$\phi \to \phi' = \lambda\phi \tag{2.6}$$

$$T_{ij} \to T'_{ij} \tag{2.7}$$

† Our notation is such that the metric has signature +2, a comma denotes partial, a semicolon covariant, differentiation, and $c = 1$.

where λ is a function of the coordinates and as yet we do not specify the form of the transformed stress–energy tensor. We need only discuss the case where λ is positive, thus maintaining an attractive gravitation. We pick our T'_{ij} in such a way that the equations for the primed quantities are precisely the same as (2.4) and (2.5) with the geometry unchanged. Assuming that this has been done, substitute (2.6) into the primed version of (2.4), expand and use (2.4) to eliminate the Einstein tensor. The result is

$$T'_{ij} = \lambda T_{ij} - (8\pi)^{-1}\phi\left(\frac{\omega}{\lambda}(\lambda_{,i}\lambda_{,j} - \frac{1}{2}g_{ij}\lambda_{,k}\lambda^{,k}) + (\lambda_{;ij} - g_{ij}\Box\lambda) + \frac{2(1+\omega)}{\phi}\lambda_{(,i}\phi_{,j)} - \frac{\omega+2}{\phi}g_{ij}\lambda_{,k}\phi^{,k}\right) \tag{2.8}$$

where a bracket around subscripts indicates the symmetric part. To ensure the conservation of the new stress–energy tensor we require

$$\Box\phi' = \frac{8\pi}{3+2\omega}T'. \tag{2.9}$$

This gives us the condition for the allowed units transformations, which is

$$\omega(\Box\mu + \mu_{,k}\psi^{,k}) = 0 \tag{2.10}$$

where $\mu = \lambda^{1/2}$ and $\psi = \ln\phi$. It should be pointed out that the units transformation considered here differs from the one discussed by Dicke (1962) who assumed that length, time and reciprocal mass all scaled in the same way. After his transformation the form of the equations (2.4) and (2.5) was altered, the scalar field lost its property as a 'geometrical' entity and test particles no longer moved along geodesics. In a recent paper Anderson (1971) has shown that the BD theory is invariant under such (conformal) transformations only if

$$\omega = -\tfrac{3}{2} \qquad T = 0 \tag{2.11}$$

in which case the equations can, by a suitable choice of scale, be put into the usual Einstein form.

The situation is different for the transformation (2.6), (2.7). From (2.10) we can see that the BD theory is form invariant under arbitrary space–time dependent changes in the unit of mass only if

$$\omega = 0. \tag{2.12}$$

T does not necessarily vanish, but R does (as can be seen by contracting (2.4)). In this case we can choose the 'gauge' $\lambda = \phi^{-1}$ in which (2.4) and (2.5) reduce to the Einstein equations (2.1) with $T' = 0$. In the case of arbitrary ω only those transformations are allowed for which $\lambda = \mu^2$ obeys

$$\Box\mu + \mu_{,k}\psi^{,k} = 0. \tag{2.13}$$

The closest analogy to (2.13) occurs in electromagnetism when the Lorentz gauge condition is imposed. The theory is then invariant only under the restricted gauge group $A_\mu \to A_\mu + \Lambda_{,\mu}$ where $\Box\Lambda = 0$ (Messiah 1961).

Even for the case of general ω it is sometimes possible to pick $\lambda = \phi^{-1}$ and transform to (2.1). Inserting this choice into (2.13) and comparing with (2.5) and the contracted (2.4) it turns out that the condition is

$$R = 0. \tag{2.14}$$

Conversely the Einstein equations, with $R = 0$, can be put into a form that is invariant under arbitrary gauge transformations of the kind (2.6), (2.7).

3. Equivalent solutions

It may be instructive to look at the above results in a different way. The general philosophy and method in BD theory is as follows: given an energy distribution T_{ij}, one is to calculate the metric g_{ij} and the gravitational 'constant' ϕ^{-1}. For the moment we will ignore Synge's (1966) objections. We have shown above that the same geometry can correspond to many solutions (ϕ, T_{ij}). We define as equivalent solutions which, for constant ω, lead to the same geometry. It is a simple matter to prove that the set of such solutions, connected by (2.6), (2.7) and obeying the constraint (2.13) form an equivalence class.

One could argue that the stress–energy tensor (2.8) is inadmissible as a solution of the BD equations because it is a function of the scalar field ϕ'. However, this only seems so because of the construction. For the new solution (ϕ', T'_{ij}) obeys the equations (2.4) and (2.5) and can be derived from an action principle (BD)

$$\delta \int \{\phi' R + 16\pi L' - \omega\phi'^{-1}(\phi'_{,i}\phi'^{,i})\}(-g)^{1/2}\, \mathrm{d}^4 x = 0 \tag{3.1}$$

in which L', the Lagrangian for the new matter field is independent of ϕ'. We could equally well have started with this solution and transformed

$$\phi' \to \phi = \lambda^{-1}\phi' \tag{3.2}$$

to the original energy tensor T_{ij}, which would then seem to be a function of ϕ. In the next section we will show some examples which demonstrate this ambiguity as to which solution represents the 'real' situation.

The above transformations can also be considered as a means of obtaining new solutions from old. There is, however, no guarantee that the transformed T_{ij} will be physically acceptable, for example that it will have a positive energy density. We will show that the same geometry can correspond to both physical and unphysical solutions.

4. Applications

4.1. Static spherical symmetry

We assume that the metric is of the form (in isotropic coordinates)

$$\mathrm{d}s^2 = -\mathrm{e}^{2\alpha}\,\mathrm{d}t^2 + \mathrm{e}^{2\beta}(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2) \tag{4.1}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$. We assume also that the scalar field ϕ and the stress–energy tensor T_{ij} are functions only of r. For the purposes of this example we consider transformations of the type (2.6), (2.7), where λ is a function of r. The new energy–

momentum tensor is then given by

$$
\begin{aligned}
T'^4_4 &= \lambda T^4_4 + h\left\{(\omega+1)\left(\frac{1}{2}\frac{\lambda_1}{\lambda}+\frac{\phi_1}{\phi}\right)-\alpha_1\right\}\\
T'^1_1 &= \lambda T^1_1 + h\left\{-\omega\left(\frac{1}{2}\frac{\lambda_1}{\lambda}+\frac{\phi_1}{\phi}\right)+\alpha_1+2\beta_1+\frac{2}{r}\right\}\\
T'^2_2 &= \lambda T^2_2 + h\left\{(\omega+1)\left(\frac{1}{2}\frac{\lambda_1}{\lambda}+\frac{\phi_1}{\phi}\right)-\beta_1-\frac{1}{r}\right\}\\
T'^3_3 &= T'^2_2
\end{aligned} \tag{4.2}
$$

where the remaining components vanish, $h = (8\pi)^{-1}\lambda_1\phi\exp(-2\beta)$, and a subscript 1 indicates differentiation with respect to r. All other components vanish because of the symmetry. For $\omega \neq 0, \lambda = \mu^2$ is restricted to being a solution of the equation

$$\mu_{11}+(2r^{-1}+\alpha_1+\beta_1+\phi^{-1}\phi_1)\mu_1 = 0. \tag{4.3}$$

This can be integrated once to yield

$$\mu_1 = kr^{-2}\phi^{-1}\exp(-\alpha-\beta) \tag{4.4}$$

where k is an arbitrary constant.

As a particular case we consider the Brans (1962) solution for the vacuum. Here we have

$$
\begin{aligned}
\exp\alpha &= \exp(\alpha_0)\left(\frac{r-B}{r+B}\right)^{1/\tau}\\
\exp\beta &= \exp(\beta_0)\left(1+\frac{B}{r}\right)^2\left(\frac{r-B}{r+B}\right)^{(\tau-c-1)/\tau}\\
\phi &= \phi_0\left(\frac{r-B}{r+B}\right)^{c/\tau}
\end{aligned} \tag{4.5}
$$

$$T_{ij} = 0 \tag{4.6}$$

where $\tau = (c^2+c+1+\frac{1}{2}\omega c^2)^{1/2}$, and $\alpha_0, \beta_0, \phi_0, B, c$ are arbitrary constants. (4.4) can now be integrated to yield

$$\mu = (2B\phi_0)^{-1}k\exp\{-(\alpha_0+\beta_0)\}\ln\left\{k_0\left(\frac{r-B}{r+B}\right)\right\} \tag{4.7}$$

where k_0 is another constant of integration. Substituting these results into (4.2) we find

$$
\begin{aligned}
T'^4_4 &= h[2B(r^2-B^2)^{-1}\{2(\omega+1)l+\tau^{-1}(c\omega+c-1)\}]\\
T'^1_1 &= h[2B(r^2-B^2)^{-1}\{-2\omega l+\tau^{-1}(c\omega+2c+1-2\tau)\}\\
&\qquad -2r^{-1}+4(r+B)^{-1}]\\
T'^2_2 &= h[2B(r^2-B^2)^{-1}\{2(\omega+1)l+\tau^{-1}(c\omega+2c+1-\tau)\}\\
&\qquad +r^{-1}-2(r+B)^{-1}]\\
T'^3_3 &= T'^2_2
\end{aligned} \tag{4.8}
$$

where

$$l = \left\{ \ln k_0 \left(\frac{r-B}{r+B} \right) \right\}^{-1}$$

$$hl = \phi_0 B^{-1} \exp(-2\beta_0) r^4 (r+B)^{-(3c+3\tau+2)/\tau} (r-B)^{(3c-3\tau+2)/\tau}$$

and we have chosen the constant k to be

$$k = 2B\phi_0 \exp\{-(\alpha_0+\beta_0)\}.$$

This is allowed since it represents a (trivial) constant scale change. The new scalar field becomes

$$\phi' = l^{-2}\phi. \tag{4.9}$$

It is interesting to examine the asymptotic behaviour of the solution (4.8), (4.9). If $k_0 \neq 1$ then as $r \to \infty$ $\phi' \to$ a constant times ϕ. The energy density $-T_4^4$ goes to zero as r^{-4} and the pressures as r^{-3}. This is unphysical. On the other hand if $k_0 = 1$ then

$$\phi' \to 4B^2\phi_0 r^{-2} \to 0 \tag{4.10}$$

and the $T^i{}_j \to 0$ all in the same way as r^{-4}. This behaviour is in keeping with the boundary conditions proposed in BD. Unfortunately the energy density in this case is negative.

For $r \to B$ the $T^i{}_j$ become infinite. This, however, may be due to the fact that the metric is singular at this surface, and the situation could possibly be altered by a coordinate transformation.

4.2. *Cosmology*

Consider a space–time with the Friedmann metric

$$ds^2 = -dt^2 + a^2(t)\{(1-kr^2)^{-1}\,dr^2 + r^2\,d\Omega^2\}. \tag{4.11}$$

We assume that our solution (ϕ, T_{ij}) represents a perfect fluid expressed in comoving coordinates, that is

$$T_{ij} = -(p+\rho)u_i u_j + pg_{ij} \tag{4.12}$$

where ρ is the energy density and p the pressure. Furthermore we assume a uniform distribution with ρ, p and ϕ functions only of t. That this does not follow from the metric, as it does in general relativity, has been shown by O'Hanlon and Tupper (1972). Nevertheless these are the solutions chosen by Brans and Dicke (BD) to represent the actual universe. We consider the transformation

$$\phi' = \lambda(t)\phi \tag{4.13}$$

that is, we assume that ϕ' is also uniform. The equation for $\lambda = \mu^2$ becomes, from (2.13), after integrating once

$$\dot{\mu} = \kappa\phi^{-1}a^{-3} \tag{4.14}$$

where κ is a constant and a dot indicates differentiation with respect to t. From (4.11), (4.12) and (4.13) it follows that the new stress–energy tensor will have as the only non-vanishing components

$$-T'^4_4 = \rho' = \lambda\rho + (8\pi)^{-1}\dot{\lambda}\phi\left\{-\omega\left(\frac{\dot{\phi}}{\phi}+\frac{1}{2}\frac{\dot{\lambda}}{\lambda}\right)+3\frac{\dot{a}}{a}\right\}$$

$$T'^1_1 = p' = \lambda p - (8\pi)^{-1}\dot{\lambda}\phi\left\{(\omega+1)\left(\frac{\dot{\phi}}{\phi}+\frac{1}{2}\frac{\dot{\lambda}}{\lambda}\right)-\frac{\dot{a}}{a}\right\}$$

$$T'^2_2 = T'^3_3 = T'^1_1 = p'. \tag{4.15}$$

Thus we have again a perfect fluid but with a different density and pressure. It is simple to check that the conservation law (2.3)

$$\dot{\rho}' + 3a^{-1}\dot{a}(\rho'+p') = 0 \tag{4.16}$$

also holds for the new fluid.

As a first example we consider the vacuum solution found previously (O'Hanlon and Tupper 1972), where

$$a = (D-kt^2)^{1/2} \tag{4.17}$$

$$\phi = ta^{-1} \tag{4.18}$$

$\omega = 0$ and all the components of T_{ij} vanish. For $k = +1$ (closed three-space) the metric (4.11) with (4.17) represents a universe which contracts to a singularity in a finite time. For $k = -1$, we have a 'bouncing' cosmology which contains no singularities. From (4.15) we find

$$\rho' = -3k(8\pi)^{-1}a^{-3}t^2\dot{\lambda}$$

$$p' = -(8\pi)^{-1}(at)^{-1}(t^2\dot{\lambda})^{\cdot} \tag{4.19}$$

when we make the transformation (4.13). Since $\omega = 0$, λ is arbitrary and need not satisfy (4.14). If we take $\lambda = t^{-1}$, p' vanishes and we get a dust solution

$$\rho' = 3k(8\pi)^{-1}a^{-3}. \tag{4.20}$$

In a similar fashion by taking

$$\lambda = (Dt)^{-1}a \tag{4.21}$$

we obtain a radiation model with density

$$\rho' = 3p' = 3k(8\pi)^{-1}a^{-4}. \tag{4.22}$$

The solutions above correspond to positive (negative) energy models if k equals $+1$ (-1). Note that (4.21) results in $\phi' =$ constant, hence (4.22) is a solution of Einstein's equations.

For arbitrary ω the only known analytic solutions for the cosmological equations are the flat-space dust solutions (BD)

$$\phi = \phi_0 t^r$$

$$a = a_0 t^q$$

$$\rho = \rho_0 t^{-3q}$$

$$p = 0$$

$$k = 0$$

$$r = 2(4+3\omega)^{-1}$$
$$q = r(1+\omega). \tag{4.23}$$

Solving (4.14) we find

$$\lambda = \kappa^2 t^{-2}(1+\kappa_1 t)^2. \tag{4.24}$$

This yields, as the new density and pressure

$$\rho' = \kappa^2 \rho_0 t^{r-4}\{\kappa_1^2 t^2 - 3(\omega+1)^2(3+2\omega)^{-1}\}$$
$$p' = -\kappa^2 \rho_0 t^{r-4}(4+3\omega)(\omega+1)(3+2\omega)^{-1}. \tag{4.25}$$

We note that the pressure is negative (for $\omega > -\frac{3}{2}$) and the density becomes positive when

$$t^2 > 3\kappa_1^{-2}(\omega+1)^2(3+2\omega)^{-1}. \tag{4.26}$$

The transformed scalar field is then

$$\phi' = \kappa^2 \phi_0 t^{r-2}(1+\kappa_1 t)^2 \tag{4.27}$$

and this is singular at $t = 0$ if $r < 2$ (which is usually the case).

5. Conclusions

We have shown that in the Brans–Dicke theory the space–time geometry determines the mass–energy content only up to a position dependent gauge transformation. Even for an attractive gravitation ($\phi > 0$) the same geometry may correspond to both positive and negative energy distributions. Thus the condition

$$R_{ij}W^iW^j \leqslant 0 \tag{5.1}$$

for $W^iW_i = 1$ a unit time-like vector, which is used to prove the singularity theorems in general relativity (Hawking and Penrose 1970), does not for the Brans–Dicke theory necessarily imply an energy condition

$$T_{ij}W^iW^j \leqslant \tfrac{1}{2}T \tag{5.2}$$

and hence these theorems need to be applied with care.

It is possible that some as yet unknown gauge condition will select out of each equivalence class the solution describing the 'real' physical situation.

The presence of the scalar field interacting with all matter equally introduces an ambiguity as to how much of the 'rest' mass is intrinsic and how much due to the scalar interaction. We have seen that not even the requirement of energy–momentum conservation can completely eradicate this ambiguity. If this question cannot be resolved then either (i) the geometrical interpretation of ϕ is spurious and the Brans–Dicke equation (2.4) can be treated as an Einstein equation where the right hand side of (2.4) is simply a peculiar stress–energy tensor (with the gauge freedom described above) or (ii) the geometry is the primary reality and the material content of a model universe becomes a question of interpretation, through a relatively arbitrary choice of the unit of mass, of this geometry. The alternative (ii) is more in keeping with the efforts of some authors to completely geometrize physics (Wheeler 1962).

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